

APPROXIMATION OF HOLOMORPHIC MAPPINGS ON STRONGLY PSEUDOCONVEX DOMAINS

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ABSTRACT. Let D be a relatively compact strongly pseudoconvex domain in a Stein manifold S . We prove that for every complex manifold Y the set $\mathcal{A}(D, Y)$ of all continuous maps $\bar{D} \rightarrow Y$ which are holomorphic in D is a complex Banach manifold, and every $f \in \mathcal{A}(D, Y)$ can be approximated uniformly on \bar{D} by maps holomorphic in open neighborhoods of \bar{D} in S . Analogous results are obtained for maps of class $\mathcal{A}^r(D)$, $r \in \mathbb{N}$. We also establish the Oka property for sections of continuous or smooth fiber bundles over \bar{D} which are holomorphic over D and whose fiber enjoys the Convex approximation property (Theorem 1.7).

1. INTRODUCTION

Let D be a relatively compact strongly pseudoconvex domain in a Stein manifold S , and let Y be a complex manifold. For $r \in \mathbb{Z}_+$ we denote by $\mathcal{C}^r(\bar{D}, Y)$ the set of all \mathcal{C}^r maps $\bar{D} \rightarrow Y$ and by $\mathcal{A}^r(D, Y)$ the set of all $f \in \mathcal{C}^r(\bar{D}, Y)$ which are holomorphic in D . Set $\mathcal{A}(D, Y) = \mathcal{A}^0(D, Y)$, $\mathcal{C}^r(\bar{D}, \mathbb{C}) = \mathcal{C}^r(\bar{D})$, $\mathcal{A}^r(D, \mathbb{C}) = \mathcal{A}^r(D)$.

In this paper we use the technique of holomorphic sprays to investigate the structure of the space $\mathcal{A}^r(D, Y)$ and to obtain approximation theorems for holomorphic maps. A *holomorphic spray* is a family of maps in a given space of maps (such as $\mathcal{A}(D, Y)$), depending holomorphically on a parameter t in an open set $P \subset \mathbb{C}^N$; the domination condition means submersivity with respect to the parameter $t \in P$ at $t = 0$ (Def. 2.1).

Theorem 1.1. *Assume that D is a relatively compact, strongly pseudoconvex domain with \mathcal{C}^ℓ boundary ($\ell \geq 2$) in a Stein manifold. For each $r \in \{0, 1, \dots, \ell\}$ and for every complex manifold Y the space $\mathcal{A}^r(D, Y)$ is a complex Banach manifold.*

Theorem 1.1 follows from Corollary 4.4 to the effect that the set of all maps in $\mathcal{A}^r(D, Y)$ which are \mathcal{C}^0 -close to a given $f_0 \in \mathcal{A}^r(D, Y)$ is isomorphic to the set of all \mathcal{C}^0 -small sections of the complex vector bundle $f_0^*(TY) \rightarrow \bar{D}$ which are holomorphic in D and of class $\mathcal{C}^r(\bar{D})$. This gives a local chart

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around $f_0 \in \mathcal{A}^r(D, Y)$ with values in the Banach space $\Gamma_{\mathcal{A}}^r(D, f_0^*(TY))$ of all sections $\bar{D} \rightarrow f_0^*(TY)$ of class $\mathcal{C}^r(\bar{D})$ which are holomorphic in D , and it is easily seen that these charts are holomorphically compatible. This Banach manifold structure is natural in the sense that the evaluation map $D \times \mathcal{A}^r(D, Y) \rightarrow Y$ is holomorphic.

For the disc $\Delta = \{z \in \mathbb{C}: |z| < 1\}$, the Banach manifold $\mathcal{A}(\Delta, Y)$ is locally modeled on the complex Banach space $\mathcal{A}(\Delta)^n$ with $n = \dim Y$. In view of the Oka-Grauert principle for vector bundles of class $\mathcal{A}(D)$, due to Leiterer [36], the same holds whenever every continuous complex vector bundle over \bar{D} is topologically trivial.

Lempert [38] obtained analogous results for the space of all \mathcal{C}^k maps from a compact \mathcal{C}^k manifold V into a complex manifold M , showing that $\mathcal{C}^k(V, M)$ is a Banach manifold if $k \in \mathbb{Z}_+$, resp. a Fréchet manifold if $k = \infty$. He also described a complex manifold structure on the space of real analytic maps by considering Stein complexifications. Like in our case, maps $f: V \rightarrow M$ near a given map f_0 correspond to \mathcal{C}^0 -small sections of the vector bundle f_0^*TM . (In this direction see also [29, 30, 32, 33]; for the corresponding theory of mappings between smooth real manifolds see Palais [44, 45].)

We now turn to holomorphic approximation theorems.

Theorem 1.2. *Assume that $D \Subset S$ is a strongly pseudoconvex domain with \mathcal{C}^ℓ boundary ($\ell \geq 2$) in a Stein manifold S , Y is a complex manifold and $r \in \{0, 1, \dots, \ell\}$. Any map in $\mathcal{A}^r(D, Y)$ can be approximated in the $\mathcal{C}^r(\bar{D}, Y)$ topology by maps holomorphic in open neighborhoods of \bar{D} .*

When $S = \mathbb{C}^n$, $Y = \mathbb{C}$ and $r = 0$, this classical result follows from the Henkin-Ramírez integral kernel representation of functions in $\mathcal{A}(D)$ [22, 31, 46, 23, p. 87]. Another approach is to cover bD by finitely many open charts in which bD is strongly convex, approximate the given function in $\mathcal{A}(D)$ by a holomorphic function in each of the charts and patch these local approximations into a global one by solving a Cousin problem with bounds. This method also works for $r > 0$ by using a solution operator to the $\bar{\partial}$ -equation for $(0, 1)$ -forms with \mathcal{C}^r estimates (Range and Siu [47], Lieb and Range [40], Michel and Perotti [41]). For approximation on certain weakly pseudoconvex domains see Fornaess and Nagel [6].

The kernel approach essentially depends on the linear structure of the target manifold and does not seem amenable to generalizations. In this paper we adapt the second approach mentioned above to an arbitrary target manifold Y by using the method of sprays. A crucial ingredient is a result from [5] concerning gluing of pairs of sprays on special configurations of domains called *Cartan pairs*. Using this gluing technique we show that any $f_0 \in \mathcal{A}^r(D, Y)$ is the core map of a dominating spray (Corollary 4.2). This immediately implies Theorem 1.1, and Theorem 1.2 is then obtained by the bumping method alluded to above. We prove the analogous approximation result for sections of holomorphic submersions (Theorem 5.1). When $r \geq 2$,

a different proof can be obtained by using the fact that the graph $G_f = \{(z, f(z)) : z \in \bar{D}\} \subset S \times Y$ of any map $f \in \mathcal{A}^2(D, Y)$ admits a basis of open Stein neighborhoods in $S \times Y$ [5, Theorem 2.6].

For approximation of holomorphic maps from planar Jordan domains to (almost) complex manifolds see D. Chakrabarti [3, Theorem 1.1.4], [4].

For maps from domains in open Riemann surfaces we obtain the following stronger result which extends Theorem 5.1 in [5].

Theorem 1.3. *Let D be a relatively compact domain with \mathcal{C}^2 boundary in an open Riemann surface S and let Y be a (reduced, paracompact) complex space. Every continuous map $\bar{D} \rightarrow Y$ which is holomorphic in D can be approximated uniformly on \bar{D} by maps which are holomorphic in open neighborhoods of \bar{D} in S . The analogous result holds for maps of class $\mathcal{A}^r(D, Y)$.*

The domains of the approximating maps in Theorems 1.2 and 1.3 must in general shrink to \bar{D} . For target manifolds Y satisfying the following *Convex approximation property* we also obtain global approximation results.

Definition 1.4. [13, Def. 1.1] A complex manifold Y enjoys the *Convex Approximation Property* (CAP) if any holomorphic map from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) to Y can be approximated uniformly on K by entire maps $\mathbb{C}^n \rightarrow Y$.

Recall that a compact set K in a complex manifold S is $\mathcal{O}(S)$ -convex if for every point $p \in S \setminus K$ there is a holomorphic function $g \in \mathcal{O}(S)$ satisfying $|g(p)| > \sup_{z \in K} |g(z)|$.

The following is a consequence of Theorem 1.2 and the main result of [13]; for a more general result see Corollary 5.3 below.

Corollary 1.5. *Assume that S is a Stein manifold and $D \Subset S$ is a strongly pseudoconvex domain with \mathcal{C}^ℓ boundary ($\ell \geq 2$) whose closure \bar{D} is $\mathcal{O}(S)$ -convex. Let $r \in \{0, 1, \dots, \ell\}$ and let $f : S \rightarrow Y$ be a \mathcal{C}^r map which is holomorphic in D . If Y enjoys CAP then f can be approximated in the $\mathcal{C}^r(\bar{D}, Y)$ topology by holomorphic maps $S \rightarrow Y$ which are homotopic to f .*

We shall now extend Corollary 1.5 to sections of certain holomorphic fiber bundles over compact strongly pseudoconvex domains.

Definition 1.6. Assume that $D \Subset S$ is domain with \mathcal{C}^ℓ boundary ($\ell \geq 2$) in a Stein manifold S , Y is a complex manifold and $r \in \{0, 1, \dots, \ell\}$. An $\mathcal{A}_Y^r(D)$ -bundle is a fiber bundle $h : X \rightarrow \bar{D}$ with fiber $h^{-1}(z) \simeq Y$ ($z \in \bar{D}$) which is smooth of class $\mathcal{C}^r(\bar{D})$ and is holomorphic over D . More precisely, every point $z_0 \in \bar{D}$ admits a relatively open neighborhood U in \bar{D} and a \mathcal{C}^r fiber bundle isomorphism $\Phi : X|_U = h^{-1}(U) \rightarrow U \times Y$ which is holomorphic over $U \cap D$.

We denote by $\Gamma^r(\bar{D}, X)$ the set of all sections $f : \bar{D} \rightarrow X$ of class \mathcal{C}^r , and by $\Gamma_{\mathcal{A}}^r(D, X)$ the set of all $f \in \Gamma^r(\bar{D}, X)$ which are holomorphic over D .

Theorem 1.7. (The Oka property for sections of $\mathcal{A}^r(D)$ -bundles) *Assume that $D \Subset S$ is a strongly pseudoconvex domain with C^ℓ boundary ($\ell \geq 2$) in a Stein manifold S and $h: X \rightarrow \bar{D}$ is an $\mathcal{A}_Y^r(D)$ -bundle for some $r \in \{0, 1, \dots, \ell\}$. If the fiber $Y = h^{-1}(z)$ ($z \in \bar{D}$) enjoys CAP then*

- (i) *every continuous section $f_0 \in \Gamma(\bar{D}, X) = \Gamma^0(\bar{D}, X)$ is homotopic to a section $f_1 \in \Gamma_{\mathcal{A}}^r(\bar{D}, X)$, and*
- (ii) *every homotopy $\{f_t\}_{t \in [0,1]} \in \Gamma(\bar{D}, X)$ with $f_0, f_1 \in \Gamma_{\mathcal{A}}^r(D, X)$ can be deformed with fixed ends to a homotopy in $\Gamma_{\mathcal{A}}^r(D, X)$.*

Theorem 1.7 is proved in §6; a more precise result, with approximation on certain compact sets in \bar{D} , is Theorem 6.1. Theorem 1.7 also applies if the base \bar{D} is a compact complex manifold with Stein interior and smooth strongly pseudoconvex boundary; such manifold is diffeomorphic to a strongly pseudoconvex domain in a Stein manifold by a diffeomorphism which is holomorphic in the interior [28, 43] (some loss of smoothness may occur). In this connection we mention Catlin's boundary version [2] of the Newlander-Nirenberg integrability theorem [42].

For fiber bundles over *open* Stein manifolds (without boundary) the corresponding result was obtained in [13] (see also [15, 21, 34]). The classical case when $X \rightarrow S$ is a principal G -bundle, with fiber a complex Lie group G , is due to H. Grauert [19, 20]. His results imply that the holomorphic classification of principal G -bundles over Stein spaces agrees with the topological classification. (See also the exposition of H. Cartan [1] and the papers [7, 8, 10, 25].) A. Sebbar [48] proved the analogous result for principal G -bundles which are smooth on the closure of a pseudoconvex domain $D \subset \mathbb{C}^n$ and holomorphic over D .

The CAP property of the fiber Y is both necessary and sufficient for validity of a stronger version of Theorem 1.7 (i) with approximation on compact holomorphically convex subsets (see Theorem 6.1 (i)). In the absence of topological obstructions, CAP of Y also implies extendibility of holomorphic maps from closed complex subvarieties in Stein manifolds to Y [11, 35]. Among the conditions implying CAP are (from strongest to weakest):

- (a) homogeneity under a complex Lie group (Grauert [19]),
- (b) the existence of a dominating spray on Y (Gromov [21]), and
- (c) the existence of a finite dominating family of sprays [9].

Further examples of manifolds enjoying CAP can be found in [12, 13].

The paper is organized as follows. In §2 we recall the notion of a (dominating) holomorphic spray and some of the related technical tools developed in [5]. A key ingredient is Proposition 2.4 concerning gluing of holomorphic sprays on Cartan pairs (Def. 2.3). In sections 3 and 4 we obtain some technical results on the existence and approximation of sprays. Theorem 1.1 is a consequence of Corollary 4.3 which provides linearization of a neighborhood of any section of class $\mathcal{A}^r(D)$. In §5 we prove Theorems 1.2 and 1.3. In §6 we prove Theorem 6.1 which includes Theorem 1.7.

2. GLUING SPRAYS ON CARTAN PAIRS

In this section D is a relatively compact, strongly pseudoconvex domain with \mathcal{C}^ℓ boundary ($\ell \geq 2$) in a Stein manifold S and $h: X \rightarrow \bar{D}$ is either an $\mathcal{A}_Y^r(D)$ -bundle for some $r \in \{0, \dots, \ell\}$, or the restriction to \bar{D} of a holomorphic submersion $\tilde{h}: \tilde{X} \rightarrow S$. (Our results also apply if $h: X \rightarrow \bar{D}$ is a smooth submersion which is holomorphic over D .)

Let $VT_x X = \ker dh_x$ denote the *vertical tangent space* at a point $x \in X$.

Definition 2.1. Assume that P is an open set in \mathbb{C}^N containing the origin. An *h -spray of class $\mathcal{A}^r(D)$ with the parameter set P* is a map $f: \bar{D} \times P \rightarrow X$ of class \mathcal{C}^r which is holomorphic on $D \times P$ and satisfies $h(f(z, t)) = z$ for all $z \in \bar{D}$ and $t \in P$. We call $f_0 = f(\cdot, 0)$ the *central* (or *core*) section of the spray. A spray f is *dominating on a subset $K \subset \bar{D}$* if the partial differential

$$\partial_t|_{t=0}f(z, t): T_0\mathbb{C}^N = \mathbb{C}^N \rightarrow VT_{f(z,0)}X$$

is surjective for all $z \in K$; if this holds with $K = \bar{D}$ then f is *dominating*.

A (dominating) *1-parametric h -spray of class $\mathcal{A}^r(D)$ and parameter set P* is a continuous map $f: [0, 1] \times \bar{D} \times P \rightarrow X$ such that for each $s \in [0, 1]$, $f^s = f(s, \cdot, \cdot): \bar{D} \times P \rightarrow X$ is a (dominating) h -spray of class $\mathcal{A}^r(D)$, and the derivatives of f^s of order $\leq r$ are continuous with respect to all variables (including s).

For a product fibration $h: X = \bar{D} \times Y \rightarrow \bar{D}$, $h(z, y) = z$, we can identify an h -spray $\bar{D} \times P \rightarrow X$ with a *spray of maps* $f: \bar{D} \times P \rightarrow Y$ by composing with the projection $\bar{D} \times Y \rightarrow Y$, $(z, y) \rightarrow y$. Such a spray is dominating on $K \subset \bar{D}$ if $\partial_t|_{t=0}f(z, t): T_0\mathbb{C}^N \rightarrow T_{f(z,0)}Y$ is surjective for all $z \in K$.

In applications the parameter set P will be allowed to shrink around $0 \in \mathbb{C}^N$. If f is dominating on a compact subset K of \bar{D} then by shrinking P we may assume that $\partial_t f(z, t): T_t\mathbb{C}^N \rightarrow VT_{f(z,t)}X$ is surjective for all $(z, t) \in K \times P$.

Remark 2.2. Our definition of a spray is similar to Def. 4.1 in [5], but is adapted for use in this paper. Sprays were introduced to the Oka-Grauert theory by Gromov [21] to obtain approximation and gluing theorems for holomorphic sections. In Gromov's terminology (which was also used in [15, 16, 17]) a dominating spray on a complex manifold Y is a holomorphic map $s: E \rightarrow Y$ from the total space of a holomorphic vector bundle $\pi: E \rightarrow Y$ such that $s(0_y) = y$ and $ds_{0_y}(E_y) = T_y Y$ for every point $y \in Y$. When $E = Y \times \mathbb{C}^N$, we get for any holomorphic map $f_0: D \rightarrow Y$ a dominating spray $f: D \times \mathbb{C}^N \rightarrow Y$ (in the sense of Def. 2.1) by setting $f(z, t) = s(f_0(z), t)$. Global dominating sprays (with the parameter set $P = \mathbb{C}^N$) exist only rarely. Sprays whose parameter set is a (small) open subset of \mathbb{C}^N have been called *local sprays* in the literature on the Oka-Grauert problem.

Definition 2.3. A pair of open subsets $D_0, D_1 \Subset S$ in a Stein manifold S is said to be a *Cartan pair* of class \mathcal{C}^ℓ ($\ell \geq 2$) if

- (i) $D_0, D_1, D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$ are strongly pseudoconvex with \mathcal{C}^ℓ boundaries, and
- (ii) $\overline{D_0 \setminus D_1} \cap \overline{D_1 \setminus D_0} = \emptyset$ (the separation property).

We say that D_1 is a *convex bump* on D_0 if, in addition to the above, there is a biholomorphic map from an open neighborhood of \bar{D}_1 in S onto an open subset of \mathbb{C}^n ($n = \dim S$) which maps D_1 and $D_{0,1}$ onto strongly convex domains in \mathbb{C}^n .

The following proposition and its proof are crucial for all that follows.

Proposition 2.4. (Gluing of sprays) *Let (D_0, D_1) be a Cartan pair of class \mathcal{C}^ℓ ($\ell \geq 2$) in a Stein manifold S and set $D = D_0 \cup D_1$, $D_{0,1} = D_0 \cap D_1$. Assume that $h: X \rightarrow \bar{D}$ is either an $\mathcal{A}_Y^r(D)$ -bundle for some $r \in \{0, \dots, \ell\}$, or the restriction to \bar{D} of a holomorphic submersion $\tilde{X} \rightarrow S$.*

Given an h -spray $f: \bar{D}_0 \times P_0 \rightarrow X$ ($P_0 \subset \mathbb{C}^N$) of class $\mathcal{A}^r(D_0)$ which is dominating on $\bar{D}_{0,1}$, there is an open set $P \subset \mathbb{C}^N$, with $0 \in P \Subset P_0$, satisfying the following. For every h -spray $f': \bar{D}_1 \times P_0 \rightarrow X$ of class $\mathcal{A}^r(D_1)$ which is sufficiently \mathcal{C}^r close to f on $\bar{D}_{0,1} \times P_0$ there exists an h -spray $g: \bar{D} \times P \rightarrow X$ of class $\mathcal{A}^r(D)$, close to f in the \mathcal{C}^r topology on $\bar{D}_0 \times P$, such that $g_0 = g(\cdot, 0)$ is homotopic to f_0 on \bar{D}_0 and is homotopic to f'_0 on \bar{D}_1 . If f and f' agree to order $m \in \mathbb{N}$ along $\bar{D}_{0,1} \times \{0\}$ then g can be chosen to agree to order m with f along $\bar{D}_0 \times \{0\}$ and with f' along $\bar{D}_1 \times \{0\}$.

If in addition $\sigma \subset \bar{D}_0$ is a common zero set of finitely many $\mathcal{A}^r(D_0)$ functions and $\sigma \cap \bar{D}_{0,1} = \emptyset$ then g can be chosen such that g_0 agrees with f_0 to a given finite order on σ .

The analogous result holds for 1-parametric sprays.

Proposition 2.4 follows from Proposition 4.3 in [5]. For later reference we recall the main steps of the proof:

Step 1: Lemma 4.4 in [5] gives a domain $P_1 \Subset P_0$ containing the origin and a *transition map* $\gamma: \bar{D}_{0,1} \times P_1 \rightarrow \mathbb{C}^N$ of class $\mathcal{A}^r(D_{0,1} \times P_1)$, close to the map $\gamma_0(z, t) = t$ in the \mathcal{C}^r topology (the closeness depending on the \mathcal{C}^r distance of f' to f on $\bar{D}_{0,1} \times P_0$) and satisfying

$$f(z, t) = f'(z, \gamma(z, t)), \quad z \in \bar{D}_{0,1}, \quad t \in P_1.$$

Step 2: Let $P \Subset P_1$ be a domain containing the origin $0 \in \mathbb{C}^N$. If γ is sufficiently \mathcal{C}^r -close to γ_0 on $\bar{D}_{0,1} \times P_1$ then Theorem 3.2 in [5] furnishes maps $\alpha: \bar{D}_0 \times P \rightarrow \mathbb{C}^N$ and $\beta: \bar{D}_1 \times P \rightarrow \mathbb{C}^N$, of class \mathcal{A}^r on their respective domains, satisfying

$$\gamma(z, \alpha(z, t)) = \beta(z, t), \quad z \in \bar{D}_{0,1}, \quad t \in P.$$

Step 3: It follows that the sprays $f(z, \alpha(z, t))$ and $f'(z, \beta(z, t))$, defined on $\bar{D}_0 \times P$ resp. on $\bar{D}_1 \times P$, amalgamate into a spray $g: \bar{D} \times P \rightarrow X$ with the stated properties. (Compare with (4.4) in [5].)

3. APPROXIMATION OF SPRAYS ON CONVEX DOMAINS

The following approximation result will be used in §4.

Lemma 3.1. *Assume that $C \subset B$ is a pair of bounded strongly convex domains with C^2 boundaries in \mathbb{C}^n , $P \subset \mathbb{C}^N$ is an open set containing the origin, and Y is a complex manifold. Given a spray of maps $f: \bar{C} \times P \rightarrow Y$ of class $\mathcal{A}^r(C, Y)$ ($r \geq 2$) whose central map $f_0 = f(\cdot, 0)$ extends to a map $\bar{B} \rightarrow Y$ of class $\mathcal{A}^r(B, Y)$, there exist*

- (i) *open sets $P_0 \supset P_1 \supset P_2 \supset \dots$ in \mathbb{C}^N , with $P_0 \subset P$ and $0 \in P_j$ for all $j \in \mathbb{Z}_+$,*
- (ii) *relatively open domains $\Omega_j \subset \bar{B} \times \mathbb{C}^N$ satisfying*

$$(\bar{C} \times \bar{P}_0) \cup (\bar{B} \times \bar{P}_j) \subset \Omega_j, \quad j = 0, 1, 2, \dots,$$

- (iii) *a sequence of maps $F_j: \Omega_j \rightarrow Y$ of class $\mathcal{A}^r(\Omega_j, Y)$ ($j \in \mathbb{Z}_+$)*

such that $F_j(\cdot, 0) = f_0$ on \bar{B} for all $j \in \mathbb{Z}_+$ and the sequence F_j converges to f in the C^r topology on $\bar{C} \times \bar{P}_0$ as $j \rightarrow \infty$.

If in addition we are given a vector bundle map $L: \bar{B} \times \mathbb{C}^N \rightarrow TY$ of class $\mathcal{A}^r(B)$ covering f_0 (i.e., such that the linear map $L_z: \{z\} \times \mathbb{C}^N \rightarrow T_{f(z,0)}Y$ is of class C^r in $z \in \bar{B}$ and is holomorphic in B) and satisfying $L_z = \partial_t f(z, t)|_{t=0}$ for $z \in \bar{C}$ then the sequence F_j can be chosen such that, in addition to the above, $\partial_t F_j(z, t)|_{t=0} = L_z$ for every $z \in \bar{B}$ and $j \in \mathbb{Z}_+$.

If $Y = \mathbb{C}^M$ then the above hold for all $r \in \mathbb{Z}_+$.

Proof. Consider first the case $Y = \mathbb{C}$. We may assume that C contains the origin $0 \in \mathbb{C}^n$ in its interior. Taylor expansion in the t variable gives

$$f(z, t) = f_0(z) + \sum_{j=1}^N g_j(z, t) t_j, \quad (z, t) \in \bar{C} \times P$$

for some $g_j \in \mathcal{A}^r(C \times P)$. For each $s < 1$ the function $g_j^s(z, t) = g_j(sz, t)$ is holomorphic in $\frac{1}{s}C \times P \supset \bar{C} \times P$; by choosing s close to one we insure that the approximation is as close as desired in $C^r(\bar{C} \times P)$. Fix an s , choose a polydisc $P_0 \Subset P$ containing $0 \in \mathbb{C}^N$ and apply Runge's theorem to approximate g_j^s in $C^r(\bar{C} \times \bar{P}_0)$ by an entire function \tilde{g}_j . The function

$$F(z, t) = f_0(z) + \sum_{j=1}^N \tilde{g}_j(z, t) t_j, \quad (z, t) \in \bar{B} \times \mathbb{C}^N$$

then approximates f on $\bar{C} \times \bar{P}_0$ and it agrees with f_0 on $\bar{B} \times \{0\}$. This gives a desired sequence F_j on the fixed domain $\Omega = \bar{B} \times P$.

Assume in addition that $L: \bar{B} \times \mathbb{C}^N \rightarrow \mathbb{C}$ is a map of class $\mathcal{A}^r(B \times \mathbb{C}^N)$ which is linear in the second variable and satisfies $L_z = \frac{\partial}{\partial t}|_{t=0} f(z, t)$ for

$z \in \bar{C}$. By Taylor's formula we have

$$f(z, t) = f_0(z) + L_z(t) + \sum_{j,k=1}^N g_{j,k}(z, t) t_j t_k, \quad (z, t) \in \bar{C} \times P,$$

for some $g_{j,k} \in \mathcal{A}^r(C \times P)$. Approximating each $g_{j,k}$ in $\mathcal{C}^r(\bar{C} \times \bar{P}_0)$ by an entire function $\tilde{g}_{j,k}: \mathbb{C}^n \times \mathbb{C}^N \rightarrow \mathbb{C}$ and setting

$$F(z, t) = f_0(z) + L_z(t) + \sum_{j,k=1}^N \tilde{g}_{j,k}(z, t) t_j t_k, \quad (z, t) \in \bar{B} \times \mathbb{C}^N,$$

gives the desired approximation.

The case $Y = \mathbb{C}^M$ follows by applying the above result to each component.

Assume now that Y is an arbitrary complex manifold and that $r \geq 2$. By [5, Theorem 2.6] the graph $\{(z, f_0(z)): z \in \bar{B}\}$ admits an open Stein neighborhood $U \subset \mathbb{C}^n \times Y$. Choose a proper holomorphic embedding $\Phi: U \rightarrow \mathbb{C}^M$, an open neighborhood $V \subset \mathbb{C}^M$ of $\Phi(U)$ and a holomorphic retraction $\iota: V \rightarrow \Phi(U)$ onto $\Phi(U)$. We apply the already proved approximation result to the map $(z, t) \rightarrow \tilde{f}(z, t) := \Phi(z, f(z, t)) \in \mathbb{C}^M$ to get a sequence \tilde{F}_j satisfying the conclusion of Lemma 3.1 with respect to \tilde{f} . Let $pr_Y: \mathbb{C}^n \times Y \rightarrow Y$ denote the projection onto the second factor. Assuming that \tilde{F}_j approximates \tilde{f} sufficiently closely on $(\bar{C} \times \bar{P}_0) \cup (\bar{B} \times \{0\})$, the latter set has an open neighborhood Ω_j on which the map $F_j = pr_Y \circ \Phi^{-1} \circ \iota \circ \tilde{F}_j$ is defined, and the resulting sequence F_j satisfies the conclusion. \square

Remark 3.2. We shall use Lemma 3.1 only with $Y = \mathbb{C}^M$ and $r \in \mathbb{Z}_+$. The reason for restricting to the case $r \geq 2$ for a general Y was that, at the time of this writing, it was not known whether the graph of any $\mathcal{A}(B)$ map with values in Y admits an open Stein neighborhood; this has been proved in a subsequent publication of the second author [14, Theorem 1.2], and therefore Lemma 3.1 holds for all $r \in \mathbb{Z}_+$ and an arbitrary manifold Y .

4. LINEARIZATION AROUND A SECTION

We shall prove that each section of class $\mathcal{A}^r(D)$ over a strongly pseudoconvex domain is the core of a dominating spray (Corollary 4.2). This is the key to all main results of the paper. Applying Cartan's Theorem B for vector bundles of class $\mathcal{A}^r(D)$ we then obtain an up to the boundary version of Grauert's tubular neighborhood theorem (Corollary 4.3), and Theorem 1.1 easily follows. The main step is provided by the following result obtained by combining Propositions 2.4 and 3.1. (Compare with [5, Lemma 4.2].)

Proposition 4.1. *Assume that D is a relatively compact strongly pseudoconvex domain with \mathcal{C}^ℓ boundary ($\ell \geq 2$) in a Stein manifold S , $r \in \{0, 1, \dots, \ell\}$, and $h: X \rightarrow \bar{D}$ is either an $\mathcal{A}_Y^r(D)$ -bundle (Def. 1.6) or the restriction to \bar{D} of a holomorphic submersion $\tilde{h}: \tilde{X} \rightarrow S$. Given a section*

$f_0 \in \Gamma_{\mathcal{A}}^r(D, X)$ and a surjective complex vector bundle map $L: \bar{D} \times \mathbb{C}^N \rightarrow VT(X)|_{f_0(\bar{D})}$ of class $\mathcal{A}^r(D)$ which covers f_0 , there exist a domain $P \subset \mathbb{C}^N$ containing the origin and a (dominating) h -spray $f: \bar{D} \times P \rightarrow X$ of class $\mathcal{A}^r(D)$ satisfying

$$(4.1) \quad f(z, 0) = f_0(z), \quad \partial_t|_{t=0}f(z, t) = L_z, \quad z \in \bar{D}.$$

Furthermore, given a homotopy of sections $f_0^s \in \Gamma_{\mathcal{A}}^r(D, X)$ ($s \in [0, 1]$), covered by a homotopy of surjective complex vector bundle maps $L^s: \bar{D} \times \mathbb{C}^N \rightarrow VT(X)|_{f_0^s(\bar{D})}$ which are holomorphic over D such that for $s = 0, 1$ the map f_0^s is the central map of a spray f^s over $\bar{D} \times P$ and $\partial_t|_{t=0}f^s(z, t) = L_z^s$ ($z \in \bar{D}$), there exist a domain $P_1 \subset P$ containing the origin and a 1-parametric spray $F: [0, 1] \times \bar{D} \times P_1 \rightarrow X$ of class $\mathcal{A}^r(D)$ such that $F^s = F(s, \cdot, \cdot)$ agrees with f^s on $\bar{D} \times P_1$ for $s = 0, 1$ and

$$(4.2) \quad F^s(z, 0) = f_0^s(z), \quad \partial_t|_{t=0}F^s(z, t) = L_z^s, \quad z \in \bar{D}, \quad s \in [0, 1].$$

If f_0^s and L^s are smooth of order k with respect to the parameter $s \in [0, 1]$ then F^s may also be chosen smooth of order k in s .

Proof. The conditions on $h: X \rightarrow \bar{D}$ imply that each point $x_0 \in X$ admits an open neighborhood $W \subset X$ isomorphic to a product $U \times V$, where U is a relatively open subset of \bar{D} and V is an open subset of a Euclidean space \mathbb{C}^l , such that in the coordinates $x = (z, w)$ ($z \in U, w \in \mathbb{C}^l$), h is the projection $(z, w) \rightarrow z$. Such coordinate neighborhoods in X will be called *special*.

By [24, Lemma 12.3] there exist strongly pseudoconvex domains $D_0 \subset D_1 \subset \dots \subset D_m = D$ with \mathcal{C}^ℓ boundaries such that $\bar{D}_0 \subset D$, and for every $j = 0, 1, \dots, m-1$ we have $D_{j+1} = D_j \cup B_j$ where B_j is a convex bump on D_j (Def. 2.3). Each of the sets B_j may be chosen so small that $f_0(B_j)$ is contained in a special coordinate neighborhood of X . The essential ingredient in the proof is Narasimhan's lemma on local convexification.

By Lemma 5.3 in [15] there exists a dominating h -spray f with core f_0 over a neighborhood of \bar{D}_0 . We recall the main idea of the proof and show that f can be chosen to satisfy $\partial_t|_{t=0}f(z, t) = L_z$ for $z \in \bar{D}_0$. Let $\{e_j\}_{j=1}^N$ be the standard basis of \mathbb{C}^N . Set

$$L_j(f_0(z)) := L_z e_j \in VT_{f_0(z)}X, \quad j = 1, \dots, N.$$

Note that $f_0(D)$ is a closed complex submanifold of $X|_D = h^{-1}(D)$ and hence admits an open Stein neighborhood $\Omega \subset X|_D$ [49]. Each L_j is a holomorphic section of the vertical tangent bundle $VT(X)$ on the submanifold $f_0(D)$ of Ω , and by Cartan's Theorem B it extends to a holomorphic vertical vector field on Ω . Denote by θ_t^j its flow. The map

$$f(z, t_1, \dots, t_N) = \theta_{t_N}^N \circ \dots \circ \theta_{t_2}^2 \circ \theta_{t_1}^1(f_0(z)),$$

which is well defined and holomorphic for z in a neighborhood of \bar{D}_0 and for $t = (t_1, \dots, t_N)$ in an open set $P \subset \mathbb{C}^N$ containing the origin, is then a dominating spray satisfying $f(\cdot, 0) = f_0$ and $\partial_t|_{t=0}f(z, t) = L_z$.

To find a desired spray on \bar{D} we perform a stepwise extension of f over the convex bumps B_0, \dots, B_{m-1} . At the j -th step we assume that we have a spray $\bar{D}_j \times P_j \rightarrow X$ with the required properties, and we shall approximate it by a spray $\bar{D}_{j+1} \times P_{j+1} \rightarrow X$ with a possibly smaller parameter set $0 \in P_{j+1} \subset P_j$. Since all steps are of the same kind, it suffices to explain the first step $j = 0$.

Applying Lemma 3.1 with the sets $C = D_0 \cap B_0$, $B = B_0$ (which are strictly convex in local holomorphic coordinates) and $Y = \mathbb{C}^M$ (since $f_0(\bar{B}_0)$ is contained in a special coordinate chart of X), we find an open set $0 \in P' \subset P_0$, a relatively open set $\Omega \subset \bar{B} \times \mathbb{C}^N$ containing $(\bar{C} \times \bar{P}_0) \cup (\bar{B} \times \bar{P}')$, and a map $f': \Omega \rightarrow \mathbb{C}^M$ of class $\mathcal{A}^r(\Omega)$ which approximates f in the \mathcal{C}^r topology on $\bar{C} \times P_0$, such that $f(z, 0) = f'(z, 0)$ and $\partial_t|_{t=0}f(z, t) = \partial_t|_{t=0}f'(z, t)$ for $z \in \bar{C}$. If the approximation is sufficiently close then Lemma 4.4 in [5] furnishes a map $\gamma(z, t) = t + c(z, t)$ of class $\mathcal{A}^r(C \times P')$ in a smaller parameter set $0 \in P' \subseteq P_0$ which is \mathcal{C}^r close to $\gamma_0(z, t) = t$, $c(z, t)$ vanishes to second order at $t = 0$ for all $z \in \bar{C}$, and $f(z, t) = f'(z, \gamma(z, t))$ for $(z, t) \in \bar{C} \times P'$.

Applying [5, Theorem 3.2] to γ on the Cartan pair (D_0, B_0) we obtain a smaller parameter set $P_1 \subset P'$ and maps $\alpha(z, t) = t + a(z, t)$ on $\bar{D}_0 \times P_1$, $\beta(z, t) = t + b(z, t)$ on $\bar{B}_0 \times P_1$, of class \mathcal{A}^r and close to $\gamma_0(z, t) = t$ on their respective domains, such that $a(z, t)$ and $b(z, t)$ vanish to second order at $t = 0$ and $\gamma(z, \alpha(z, t)) = \beta(z, t)$ for $z \in \bar{C}$ and $t \in P_1$. (See Step (ii) in the proof of Proposition 2.4 above.) Now $(z, t) \rightarrow f(z, \alpha(z, t))$ is a spray of class $\mathcal{A}^r(D_0)$ which is \mathcal{C}^r -close to f and agrees with f to second order at $t = 0$, $f'(z, \beta(z, t))$ is a spray with the analogous properties on $\bar{B}_0 \times P_1$, and by construction the two sprays agree on $\bar{C} \times P_1$; hence they define a spray satisfying (4.1) on the set $\bar{D}_1 = \bar{D}_0 \cup \bar{B}_0$. After m steps of this kind we obtain the first part of the Proposition.

We continue with the parametric case (this will only be used in the proof of Theorem 1.7). Fix an $s \in [0, 1]$. By the first part of the Proposition there exists an h -spray $f^s: \bar{D} \times P \rightarrow X$ of class $\mathcal{A}^r(D)$ satisfying

$$f^s(z, 0) = f_0^s(z), \quad \partial_t|_{t=0}f^s(z, t) = L_z^s, \quad z \in \bar{D}.$$

For $s = 0, 1$ we use the already given sprays. We wish to choose these sprays to depend continuously (or smoothly) on the parameter s . To do this, we shall first use a fixed spray f^s to find a solution in an open interval $I_s \subset \mathbb{R}$ around s , and finally we shall patch these solutions together.

Fix a number $u \in [0, 1]$. Since L^u is surjective, there is a direct sum splitting $\bar{D} \times \mathbb{C}^N = E \oplus G$, where E and G are vector bundles of class $\mathcal{A}^r(D)$ and $E_z = \ker L_z^u$ for each $z \in \bar{D}$. (We use Theorem B for $\mathcal{A}(D)$ -bundles, due to Leiterer [37], and Heunemann's approximation theorem [26]; compare with the proof of Lemma 4.4 and the Appendix in [5].) We split the fiber variable on $\{z\} \times \mathbb{C}^N$ accordingly as $t = t'_z \oplus t''_z \in E_z \oplus G_z$. Note that $L^u: G \rightarrow VT(X)|_{f^u(\bar{D})}$ is a complex vector bundle isomorphism of class $\mathcal{A}^r(D)$. By the inverse function theorem there is an open interval

$I_u = (u - \delta, u + \delta) \subset \mathbb{R}$ such that for each $s \in I_u \cap [0, 1]$ there exists a unique section $g_s: \bar{D} \rightarrow G$ of class $\mathcal{A}^r(D)$ satisfying

$$f^u(z, 0'_z \oplus g_s(z)) = f_0^s(z), \quad z \in \bar{D}.$$

It follows that the map

$$H^s(z, t) = f^u(z, t'_z \oplus (t''_z + g_s(z)))$$

is a dominating 1-parametric spray with the core f_0^s for $s \in I_u \cap [0, 1]$. For $s = u$ we have $g_u = 0$ and $H^u = f^u$.

It remains to adjust the t -differential of H^s at $t = 0$. Elementary linear algebra shows that, after shrinking the interval I_u if necessary, there exist for every $s \in I_u$ a unique complex vector bundle automorphism $A^s: G \rightarrow G$ and a unique complex vector bundle map $B^s: E \rightarrow G$, both of class $\mathcal{A}^r(D)$ and continuous in s , such that the map

$$F^s(z, t) = H^s(z, t'_z \oplus (B_z^s t'_z + A_z^s t''_z)), \quad s \in I_u \cap [0, 1],$$

is a 1-parametric spray satisfying $\partial_t|_{t=0} F^s(z, t) = L_z^s(t)$ for all $z \in \bar{D}$ and $s \in I_u \cap [0, 1]$.

The above argument gives a finite covering of $[0, 1]$ by intervals $I_j = [a_j, b_j]$ ($j = 0, 1, \dots, m$), where $a_0 = 0 < a_1 < b_0 < a_2 < b_1 < \dots < b_m = 1$, and for each j a 1-parametric spray $F_j: I_j \times \bar{D} \times P \rightarrow X$ satisfying the conclusion of the Proposition on I_j .

It remains to patch the sprays $\{F_j\}_{j=0}^m$ into a 1-parametric spray F on $[0, 1]$. Since each pair of adjacent intervals I_j, I_{j+1} intersect in the segment $[a_{j+1}, b_j]$ while each three intervals are disjoint, it suffices to explain how to patch F_j and F_{j+1} to a spray over $I_j \cup I_{j+1}$. Choose a point $u \in (a_{j+1}, b_j)$. Using a decomposition $\bar{D} \times \mathbb{C}^N = E \oplus G$ as above, with $E = \ker \partial_t|_{t=0} F_j(u, \cdot, t)$, the implicit function theorem gives a segment $J_j = [\alpha_j, \beta_j] \subset (a_{j+1}, b_j)$ with $\alpha_j < u < \beta_j$, a polydisc $P_0 \subset P$ containing $0 \in \mathbb{C}^N$, and for each $s \in J_j$ a unique map $\gamma(s, z, t) = t'_z \oplus (t''_z + c(s, z, t))$ of class $\mathcal{A}^r(D \times P_0)$ whose derivatives of order $\leq r$ in $(z, t) \in \bar{D} \times P_0$ are continuous in all variables and $c(s, z, t)$ vanishes to second order at $t = 0$, such that

$$F_j(s, z, \gamma(s, z, t)) = F_{j+1}(s, z, t), \quad s \in I_j, \quad z \in \bar{D}, \quad t \in P_0.$$

(The special form of γ is insured by the fact that the first order jets of the sprays F_j and F_{j+1} with respect to t agree at $t = 0$ for every $s \in I_j$. For more details see Lemma 4.4 in [5].) Choose a smooth function $\chi: \mathbb{R} \rightarrow [0, 1]$ such that $\chi(s) = 0$ for $s \leq \alpha_j$ and $\chi(s) = 1$ for $s \geq \beta_j$. The map

$$(s, z, t) \mapsto F_j(s, z, t'_z \oplus (t''_z + \chi(s)c(s, z, t)))$$

is then a 1-parametric spray of class $\mathcal{A}^r(D)$ satisfying the Proposition on $I_j \cup I_{j+1}$. After m steps we obtain a solution on $[0, 1]$. \square

A map L as in Proposition 4.1 always exists for a sufficiently large N as follows from Cartan's Theorem A for coherent sheaves of \mathcal{A}^r modules on strongly pseudoconvex domains ([27, Theorem 6], [40]). Hence we get

Corollary 4.2. (Existence of dominating sprays) *Given $h: X \rightarrow \bar{D}$ and a section $f_0 \in \Gamma_{\mathcal{A}}^r(D, X)$ as in Proposition 4.1, there exist a domain $0 \in P \subset \mathbb{C}^N$ for some $N \in \mathbb{N}$ and a dominating h -spray $f: \bar{D} \times P \rightarrow X$ of class $\mathcal{A}^r(D)$ with the central section f_0 . Furthermore, for any homotopy of sections $f_0^s \in \Gamma_{\mathcal{A}}^r(D, X)$ ($s \in [0, 1]$) there exists a homotopy of dominating h -sprays $f^s: \bar{D} \times P \rightarrow X$ of class $\mathcal{A}^r(D)$ such that the core of f^s equals f_0^s for every $s \in [0, 1]$.*

Corollary 4.2 implies the following ‘up to the boundary’ version of Grauert’s tubular neighborhood theorem which allows linearization of analytic problems near a given section.

Corollary 4.3. (Linearizing a neighborhood) *Let $h: X \rightarrow \bar{D}$ be as in Proposition 4.1. Given a section $f \in \Gamma_{\mathcal{A}}^r(D, X)$ there exist a holomorphic vector bundle $\pi: E \rightarrow \bar{D}$ over an open neighborhood $\tilde{D} \subset S$ of \bar{D} , a relatively open neighborhood Ω of the zero section of the restricted bundle $E|_{\bar{D}} := \pi^{-1}(\bar{D})$ and a fiber preserving \mathcal{C}^r diffeomorphism $\Phi: \Omega \rightarrow \Phi(\Omega) \subset X$ which is biholomorphic on $\Omega \cap \pi^{-1}(D)$ and maps the zero section of $E|_{\bar{D}}$ onto $f(\bar{D})$.*

Proof. Let $f: \bar{D} \times P \rightarrow X$ be a dominating spray furnished by Corollary 4.2. There is a splitting $\bar{D} \times \mathbb{C}^N = E \oplus E'$ of $\mathcal{A}^r(D)$ -vector bundles, where $E'_z = \ker \partial_t|_{t=0} f(z, t)$ and E is a complementary bundle (Theorem B for \mathcal{A}^r bundles, [27]). The restriction of f to $\Omega = E \cap P$ satisfies Corollary 4.3. \square

Note that E in Corollary 4.3 is just the normal bundle of the given section $f: \bar{D} \rightarrow X$, and it can be identified with the restriction of the vertical tangent bundle $VT(X) = \ker dh$ to $f(\bar{D})$. When $r \geq 2$ and h extends to a holomorphic submersion $\tilde{X} \rightarrow \tilde{D}$ over an open neighborhood \tilde{D} of \bar{D} in S , $f(\bar{D})$ admits a basis of open Stein neighborhoods in \tilde{X} [5, Theorem 2.6] and the conclusion of Corollary 4.3 easily follows from standard methods.

Corollary 4.3 implies the following result concerning the deformation space of a map in $\mathcal{A}^r(D, Y)$. For related results concerning maps in L^2 -Sobolev classes from certain bordered Riemann surfaces to (almost) complex manifolds see Ivashkovich and Shevchishin [29, 30].

Corollary 4.4. (The deformation space of an \mathcal{A}^r map) *Assume that D is a relatively compact domain with strongly pseudoconvex boundary of class \mathcal{C}^ℓ ($\ell \geq 2$) in a Stein manifold. If $r \in \{0, 1, \dots, \ell\}$ and Y is an arbitrary complex manifold then for any $f_0 \in \mathcal{A}^r(D, Y)$ the space of all $f \in \mathcal{A}^r(D, Y)$ which are sufficiently \mathcal{C}^0 -close to f_0 is isomorphic to the space of \mathcal{C}^0 -small sections in $\Gamma_{\mathcal{A}}^r(D, E)$, where $E = f_0^*(TY)$.*

Proof. We may consider maps $f: \bar{D} \rightarrow Y$ as sections of the product fibration $h: X = \bar{D} \times Y \rightarrow \bar{D}$. Fix $f_0 \in \Gamma_{\mathcal{A}}^r(D, X)$. Let $\Phi: \Omega \rightarrow \Phi(\Omega) \subset X$ be as in Corollary 4.3, where Ω is an open neighborhood of the zero section in the complex vector bundle $E = f_0^*(TY)$ and Φ maps the zero section of E onto $f_0(\bar{D})$. If $f: \bar{D} \rightarrow X$ is a section in $\Gamma_{\mathcal{A}}^r(D, X)$ which is sufficiently uniformly close to f_0 then $f(\bar{D}) \subset \Phi(\Omega)$, and hence $f = \Phi \circ f'$ for a unique $f' \in \Gamma_{\mathcal{A}}^r(D, E)$ with $f'(\bar{D}) \subset \Omega$. \square

Proof of Theorem 1.1. The proof is similar to Lempert's construction in [38, §2], and our Corollary 4.4 plays a similar role as Lemma 2.1 in that paper. (For the smooth case see also Palais [44, 45].) The key difference is that the sprays in our paper must be holomorphic in the base variable $z \in D$, and their construction is one of the main technical ingredients of our proof.

Given $f_0 \in \mathcal{A}^r(D, Y)$, Corollary 4.4 furnishes a contractible local chart $\mathcal{U} \subset \mathcal{A}^r(D, Y)$ consisting of all maps $z \rightarrow \Phi(z, \xi(z))$ ($z \in \bar{D}$), where $\xi \in \Gamma_{\mathcal{A}}^r(D, f_0^*(TY))$ is a section with range in an open, fiberwise contractible set $\Omega_0 \subset E_0 = f_0^*(TY)$ containing the zero section. The transition map between any pair of such charts is of the form

$$\Gamma_{\mathcal{A}}^r(D, E_0) \ni \xi \rightarrow \tilde{\xi} \in \Gamma_{\mathcal{A}}^r(D, E_1),$$

where $\tilde{\xi}(z) = \Psi(z, \xi(z))$ ($z \in \bar{D}$) for a fiber preserving diffeomorphism Ψ of class \mathcal{A}^r from an open subset of $\Omega_0 \subset E_0$ onto an open subset of $E_1 = f_1^*(TY)$. (Actually E_0 and E_1 are isomorphic since f_1 is isotopic to f_0 due to fiber contractibility of the set Ω_0 .) In local coordinates (z, w) on E_0 resp. E_1 , Ψ is of the form $\Psi(z, w) = (z, \psi(z, w))$. The differential of the transition map $\xi \rightarrow \tilde{\xi}$ at ξ_0 , applied to a tangent vector $\eta \in \Gamma_{\mathcal{A}}^r(D, E_0)$, equals

$$z \rightarrow \partial_2 \Psi(z, \xi(z)) \eta(z), \quad z \in \bar{D}.$$

Since the partial differential $\partial_2 \Psi$ is nondegenerate and the \mathcal{C}^r regularity up to the boundary is preserved when differentiating on the fiber variable, we see that the differential is a complex Banach space isomorphism effected by $\partial_2 \Psi(\cdot, \xi(\cdot))$, and hence the transition map is biholomorphic. \square

5. UNIFORM APPROXIMATION OF HOLOMORPHIC SECTIONS

In this section we prove the following approximation theorem which includes Theorem 1.2 as a special case.

Theorem 5.1. *Assume that $D \Subset S$ is a strongly pseudoconvex domain of class \mathcal{C}^ℓ ($\ell \geq 2$) in a Stein manifold S . Let $h: X \rightarrow S$ be a holomorphic submersion of a complex manifold X onto S and let $r \in \{0, 1, \dots, \ell\}$. Every section $f \in \Gamma_{\mathcal{A}}^r(D, X)$ can be approximated in the \mathcal{C}^r topology on \bar{D} by sections which are holomorphic in open neighborhoods of \bar{D} in S .*

Proof. Let $n = \dim S$ and $n + m = \dim X$. Denote by pr_1 resp. pr_2 the coordinate projections of $\mathbb{C}^n \times \mathbb{C}^m$ onto the respective factors \mathbb{C}^n , \mathbb{C}^m , and let $B \subset \mathbb{C}^n$, $B' \subset \mathbb{C}^m$ denote the unit balls.

Since $h: X \rightarrow S$ is a holomorphic submersion, there exist for each point $x_0 \in X$ open neighborhoods $x_0 \in W \subset X$, $h(x_0) \in V \subset S$, and biholomorphic maps $\phi: V \rightarrow B \subset \mathbb{C}^n$, $\Phi: W \rightarrow B \times B' \subset \mathbb{C}^n \times \mathbb{C}^m$, such that $\phi(h(x)) = pr_1(\Phi(x))$ for every $x \in W$. Thus Φ is of the form

$$\Phi(x) = (\phi(h(x)), \phi'(x)) \in B \times B', \quad x \in W,$$

where $\phi' = pr_2 \circ \Phi$. Let us call such (W, V, Φ) a *special coordinate chart* on X . Note that $h(W) = V$.

Fix a section $f: \bar{D} \rightarrow X$ in $\Gamma_{\mathcal{A}}^r(D, X)$. Using Narasimhan's lemma on local convexification of a strongly pseudoconvex hypersurface we find finitely many special coordinate charts (W_j, V_j, Φ_j) on X , with $\Phi_j = (\phi_j \circ h, \phi'_j)$, such that $bD \subset \bigcup_{j=1}^{j_0} V_j$ and the following hold for $j = 1, \dots, j_0$:

- (i) $\phi_j(bD \cap V_j)$ is a strongly convex hypersurface in the ball $B \subset \mathbb{C}^n$,
- (ii) $f(\bar{D} \cap V_j) \subset W_j$, and
- (iii) $\overline{\phi'_j(f(\bar{D} \cap V_j))} \subset B'$.

Choose a number $c < 1$ sufficiently close to 1 such that the sets $U_j = \phi_j^{-1}(cB) \Subset V_j$ ($j = 1, \dots, j_0$) still cover bD .

By a finite induction we shall construct an increasing sequence of strongly pseudoconvex domains with \mathcal{C}^ℓ boundaries $D = D_0 \subset D_1 \subset \dots \subset D_{j_0} \Subset S$ and sections $f_k \in \Gamma_{\mathcal{A}}^r(D_k, X)$ ($k = 0, 1, \dots, j_0$), with $f_0 = f$, such that for every $k = 1, \dots, j_0$ the restriction of f_k to \bar{D}_{k-1} will be close to the previous section f_{k-1} in $\Gamma_{\mathcal{A}}^r(\bar{D}_{k-1}, X)$. The domain D_k of f_k will in general depend on the rate of approximation on D_{k-1} and will be chosen such that

$$D_{k-1} \subset D_k \subset D_{k-1} \cup V_k, \quad bD_{k-1} \cap U_k \subset D_k$$

for $k = 1, \dots, j_0$; that is, we enlarge D_{k-1} inside the k -th coordinate neighborhood V_k so that the part of bD_{k-1} which lies in the smaller set U_k is contained in the next domain D_k . The final domain D_{j_0} will contain \bar{D} in its interior, and the section $f_{j_0} \in \Gamma_{\mathcal{A}}^r(D_{j_0}, X)$ will approximate f as close as desired in $\mathcal{C}^r(\bar{D}, X)$. To keep the induction going we will also insure at every step that the properties (ii) and (iii) above remain valid with (D, f) replaced by (D_k, f_k) for all $j = 1, \dots, j_0$.

Since all steps will be of exactly the same kind, we shall explain how to get (D_1, f_1) from $(D, f) = (D_0, f_0)$. We begin by finding a domain D'_1 with \mathcal{C}^ℓ boundary in S which is a convex bump on $D = D_0$ (Def. 2.3) such that $\bar{U}_1 \cap \bar{D}_0 \subset D'_1 \Subset V_1$. We shall first find a set $\tilde{D}'_1 \Subset B$ with desired properties and then take $D'_1 = \phi_1^{-1}(\tilde{D}'_1)$.

Choose a smooth function $\chi \geq 0$ with compact support contained in $B \subset \mathbb{C}^n$ such that $\chi = 1$ on cB . Let $\tau: B \rightarrow \mathbb{R}$ be a strongly convex defining function for the domain $\phi_1(D \cap V_1) \subset B$. Choose $c' \in (c, 1)$ close

to 1 such that the hypersurface $\phi_1(bD \cap V_1) = \{\tau = 0\}$ intersects the sphere $\{|z| = c'\}$ transversely. If $\delta > 0$ is chosen sufficiently small then the set

$$\{z \in \mathbb{C}^n : |z| < c', \tau(z) < \delta\chi(z)\}$$

satisfies the required properties, except that it is not smooth along the intersection of the hypersurfaces $\{|z| = c'\}$ and $\{\tau = \delta\chi\}$. By rounding off the corners of the intersection we get a strongly convex set \tilde{D}'_1 in B such that $D'_1 = \phi_1^{-1}(\tilde{D}'_1) \subset V_1$ satisfies the desired properties (Fig. 1).

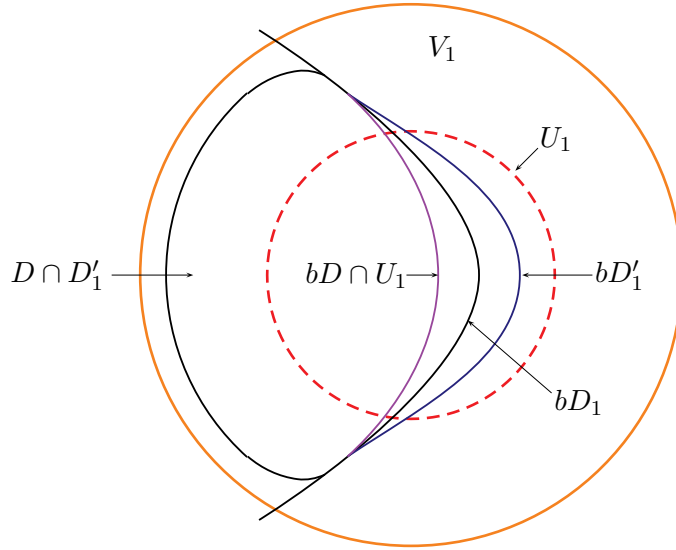


FIGURE 1. The domains D'_1 and D_1

Proposition 4.1 furnishes a dominating h -spray $F: \bar{D} \times P \rightarrow X$ of class $\mathcal{A}^r(D)$ with the core section $F_0 = f$. By shrinking its parameter set $P \subset \mathbb{C}^N$ if necessary we can insure that properties (ii) and (iii) above are satisfied if we replace $f = F_0$ by any section $F_t = F(\cdot, t)$, $t \in P$.

By using the special coordinate chart (W_1, V_1, Φ_1) we find an open set $\Omega \subset V_1$ containing $\bar{D} \cap V_1$ and a holomorphic h -spray $G: \Omega \times P \rightarrow X|_\Omega = h^{-1}(\Omega)$, with range contained in W_1 , such that the restriction of G to $(\bar{D} \cap V_1) \times P$ approximates F as close as desired in the \mathcal{C}^r topology. (The size of Ω will depend on G .)

We wish to use Proposition 2.4 to glue F and G into a single h -spray. We cannot do this directly since their domains do not form a Cartan pair. (The domain of G need not contain all of the set \bar{D}'_1 which forms a Cartan pair with D .) We proceed as follows. If G is sufficiently close to F in the \mathcal{C}^r topology on the intersection of their domains, Lemma 4.4 in [5] furnishes a transition map $\gamma(z, t) = z + c(z, t)$ of class \mathcal{A}^r between F and G , defined for $z \in \bar{D} \cap \bar{D}'_1$ and t in a smaller parameter set $P' \Subset P \subset \mathbb{C}^N$, such that

γ is \mathcal{C}^r close to $\gamma_0(z, t) = t$ (depending on the closeness of G to F), and $F(z, t) = G(z, \gamma(z, t))$ for $z \in \bar{D} \cap \bar{D}'_1$ and $t \in P'$.

Theorem 3.2 in [5], applied to γ on the Cartan pair (D, D'_1) , furnishes a smaller parameter set $P_1 \subset P'$ and maps $\alpha(z, t) = t + a(z, t)$ on $\bar{D} \times P_1$, $\beta(z, t) = t + b(z, t)$ on $\bar{D}'_1 \times P_1$, of class \mathcal{A}^r and close to γ_0 on their respective domains, such that $\gamma(z, \alpha(z, t)) = \beta(z, t)$ for $z \in \bar{D} \cap \bar{D}'_1$ and t close to 0. The spray $F(z, \alpha(z, t))$ is then defined on $\bar{D} \times P_1$ and is close to F , the spray $G(z, \beta(z, t))$ is defined on $(\bar{D}'_1 \cap \Omega) \cap P_1$, and by the construction these sprays agree for $z \in \bar{D} \cap \bar{D}'_1 \subset \Omega$. The central section f_1 of the new amalgamated spray, obtained by setting $t = 0$, is then of class \mathcal{A}^r on $\bar{D} \cup (\bar{D}'_1 \cap \Omega)$.

It remains to restrict f_1 to a suitably chosen strongly pseudoconvex domain $D_1 \Subset S$ contained in $D \cup (\bar{D}'_1 \cap \Omega)$ and satisfying the other required properties. We choose D_1 such that it agrees with D outside of V_1 , while

$$D \cap V_1 = \phi_1^{-1}(\{z \in B : \tau(z) < \epsilon \chi(z)\})$$

for a small $\epsilon > 0$ (Fig. 1). By choosing $\epsilon > 0$ sufficiently small (depending on f_1) we insure that the hypotheses (i)–(iii) are satisfied by the pair (D_1, f_1) . This completes the first step.

Applying the same procedure to (D_1, f_1) and the second chart (W_2, V_2, Φ_2) we get the next pair (D_2, f_2) . After j_0 steps we find a section f_{j_0} over a neighborhood D_{j_0} of \bar{D} in S which approximates f as close as desired in the \mathcal{C}^r topology on \bar{D} . Since all steps were made by homotopies, $f_{j_0}|_{\bar{D}}$ is homotopic to $f = f_0$ in $\Gamma_{\mathcal{A}}^r(D, X)$. \square

Proof of Theorem 1.3. Assume that D is a relatively compact connected domain with \mathcal{C}^2 boundary in an open Riemann surface S and $f_0: \bar{D} \rightarrow Y$ is a map of class $\mathcal{A}(D, Y)$ to an n -dimensional complex space Y . We wish to prove that f_0 can be uniformly approximated by maps which are holomorphic in open neighborhoods of \bar{D} in S . (The analogous result for maps of class $\mathcal{A}^r(D, Y)$ can be proved in a similar way.)

We proceed by induction on the dimension of Y . Assume that the result already holds for complex spaces of dimension $< n = \dim Y$ (this is trivially satisfied when $n = 1$). If $f_0(\bar{D}) \subset Y_{\text{sing}}$, the inductive hypothesis gives approximation of f_0 by holomorphic maps to Y_{sing} . (It may happen that the image of any nearby holomorphic map is contained in Y_{sing} ; see the example in [18].)

Suppose now that $f_0(\bar{D}) \not\subset Y_{\text{sing}}$. The set $\sigma = \{z \in \bar{D} : f_0(z) \in Y_{\text{sing}}\}$ is locally the common zero set of finitely many functions of class $\mathcal{A}(D)$; hence σ is a closed subset of \bar{D} such that $\sigma \cap D$ is discrete in D and $\sigma \cap bD$ has linear measure zero in bD .

We shall need the following result analogous to Corollary 4.2.

Lemma 5.2. *(Notation as above.) There exists a spray of maps $f: \bar{D} \times P \rightarrow Y$ of class $\mathcal{A}(D)$ which is dominating on $\bar{D} \setminus \sigma$ and whose core map is f_0 .*

Proof. As in the proof of Proposition 4.1 we choose a finite sequence of domains $D_0 \subset D_1 \subset \dots \subset D_m = D$ with \mathcal{C}^2 boundaries such that $\bar{D}_0 \subset D$, and for every $j = 0, 1, \dots, m-1$ we have $D_{j+1} = D_j \cup B_j$ where B_j is a convex bump on D_j (Def. 2.3). In addition, the properties of the set $\sigma = f_0^{-1}(Y_{sing})$ imply that the sets B_j and D_j can be chosen such that the following hold for each $j = 0, \dots, m-1$:

- (i) $\sigma \cap bD_j \cap D = \emptyset$,
- (ii) $\sigma \cap bD_j \cap bD$ is contained in the relative interior of $bD_j \cap bD$ (in bD),
- (iii) $\sigma \cap \bar{D}_j \cap \bar{B}_j = \emptyset$.

Lemma 4.2 in [5] furnishes sprays $f: \bar{D}_0 \times P \rightarrow Y$ and $f': \bar{B}_0 \times P \rightarrow Y$, satisfying the required properties over \bar{D}_0 resp. \bar{B}_0 . In particular, the core of each of these two sprays is f_0 , restricted to the respective domain, and the exceptional set (the set where the spray fails to dominate) is σ . Note that both sprays are dominating over the convex set $\bar{C}_0 = \bar{D}_0 \cap \bar{B}_0$ by (iii), and they agree at $t = 0$.

The proof of Lemma 4.4 in [5] gives a map $\gamma(z, t)$ of class $\mathcal{A}(C_0 \times P)$ (after shrinking P around 0), satisfying $f(z, t) = f'(z, \gamma(z, t))$ for $(z, t) \in \bar{C}_0 \times P$ and $\gamma(z, 0) = 0$ for $z \in \bar{C}_0$. (In that proof we have assumed that the two sprays are close to each other, but in the present situation this is not necessary since $f(z, 0) = f'(z, 0)$ for $z \in \bar{C}_0$. It suffices to find a direct summand of class $\mathcal{A}(C_0)$ in $\bar{D}_0 \times \mathbb{C}^N$ to the kernel of the t -derivative of each of the two sprays and apply the implicit function theorem as in [5].)

As in the proof of Lemma 3.1 we approximate γ uniformly on $\bar{C}_0 \times P$ by an entire map $\gamma'(z, t)$ satisfying $\gamma'(z, 0) = 0$ for $z \in \mathbb{C}^n$. The spray $(z, t) \rightarrow f'(z, \gamma'(z, t))$ is then defined on $(\bar{C}_0 \times P) \cup (\bar{B}_0 \times P_0)$ for some polydisc $P_0 \subset \mathbb{C}^n$ around 0 (which may depend on γ'), and it agrees with f for $t = 0$. If the two sprays are sufficiently uniformly close to each other on $\bar{C}_0 \times P$ (as we may assume to be the case), we can glue them into a new spray \tilde{f} with the required properties over $\bar{D}_1 = \bar{D}_0 \cup \bar{B}_0$. (See the proof of Proposition 4.1 above, or the proof of Proposition 4.3 in [5]. The important point is that our sprays agree along $t = 0$, so we get a nonempty parameter set for the new spray.) Note also that the gluing process does not increase the exceptional set of the spray. After finitely many steps of this kind we get a desired spray f over \bar{D} . \square

To complete the proof of Theorem 1.3 we proceed exactly as in the proof of Theorem 5.1 in [5]. The main idea is to attach to \bar{D} a convex bump B such that $\sigma \cap \bar{D} \cap \bar{B} = \emptyset$ (this is possible since $\sigma \cap bD$ has empty interior in bD) and then approximate f by another spray over $K = \bar{D} \cup \bar{B}$. Let $g: K \rightarrow Y$ denote the core of the new spray. There exists a holomorphic vector field ξ in a neighborhood of K in S which points into D at every point of $bD \setminus B$ (we put no condition on ξ at the points of $bD \cap \bar{B}$; see [5]). Denoting its flow by ϕ_t , the map $g \circ \phi_t$ is defined and holomorphic in an open

neighborhood of \bar{D} for every sufficiently small $t > 0$, and it approximates g (and hence f_0) uniformly on \bar{D} . \square

Combining Theorem 5.1 with the main result of [13] gives the following version of the Oka principle. Note that Corollary 1.5 is just Corollary 5.3 applied to the product submersion $X = S \times Y \rightarrow S$.

Corollary 5.3. *Assume that $D \Subset S$ is a strongly pseudoconvex domain with $\mathcal{O}(S)$ -convex closure in a Stein manifold S and $h: X \rightarrow S$ is a holomorphic fiber bundle whose fiber Y enjoys CAP (Def. 1.4). For any continuous section $f_0: S \rightarrow X$ which is holomorphic on D there exists a homotopy $f_t: S \rightarrow X$ of continuous sections such that $f_t|_D$ is holomorphic and uniformly close to $f_0|_D$ for each $t \in [0, 1]$, and f_1 is holomorphic on S .*

Furthermore, every homotopy of continuous sections $f_t: S \rightarrow X$, with $f_t|_D$ holomorphic for each $t \in [0, 1]$ and f_0, f_1 holomorphic on S , can be deformed with fixed ends to a homotopy $\tilde{f}_t: S \rightarrow X$ consisting of sections which are holomorphic on S such that the entire homotopy remains holomorphic on D .

The analogous results hold for sections of class \mathcal{C}^r provided that $bD \in \mathcal{C}^\ell$ ($\ell \geq 2$) and $r \in \{0, 1, \dots, \ell\}$.

Proof. By Theorem 5.1 we can approximate f as close as desired in the $\mathcal{C}^r(\bar{D}, X)$ topology by a \mathcal{C}^r section $g: U \rightarrow X$ which is holomorphic in an open neighborhood U of \bar{D} in S . If the approximation of f by g is sufficiently close and U is chosen sufficiently small then g is homotopic to $f|_U$ by a homotopy of \mathcal{C}^r sections $g_t: U \rightarrow X$ ($t \in [0, 1]$, $g_0 = f$, $g_1 = g$) which are holomorphic in D (Corollary 4.3). Choose a smooth function $\chi: S \rightarrow [0, 1]$ with compact support contained in U such that $\chi = 1$ in a smaller open neighborhood of \bar{D} . Set $\tilde{g}_t(z) = g_{\chi(z)t}(z)$; this is a homotopy of \mathcal{C}^r sections which are holomorphic on D , they extend to all of S and equal f on $S \setminus U$.

Assuming that the fiber Y of $h: X \rightarrow S$ enjoys CAP, Theorem 1.2 in [13] shows that \tilde{g}_1 is homotopic to a holomorphic section $f_1: S \rightarrow X$ by a homotopy of sections $h_t: S \rightarrow X$ ($t \in [0, 1]$, $h_0 = \tilde{g}_1$, $h_1 = f_1$) which are holomorphic and uniformly close to \tilde{g}_1 in an open neighborhood of \bar{D} . By combining the homotopies \tilde{g}_t and h_t we obtain a homotopy from $f = f_0$ to f_1 satisfying the stated properties.

Similarly, a homotopy $\{f_t\}_{t \in [0, 1]}$ in the second statement can be deformed (with fixed ends at $t = 0, 1$) to a homotopy $\{f'_t\}_{t \in [0, 1]}$ consisting of sections which are holomorphic in an open neighborhood of \bar{D} in S ; it remains to apply the one-parametric Oka principle [13, Theorem 5.1] to $\{f'_t\}$. \square

6. THE OKA PROPERTY FOR $\mathcal{A}^r(D)$ -BUNDLES WITH FLEXIBLE FIBERS

Let D be a relatively compact strongly pseudoconvex domain in a Stein manifold S . Recall that $\mathcal{A}(D)$ denotes the algebra of all continuous functions on \bar{D} which are holomorphic on D ; we shall use the analogous notation for (open or closed) domains in \bar{D} .

A compact set K in \bar{D} is $\mathcal{A}(D)$ -convex if for every $p \in \bar{D} \setminus K$ there is a $g \in \mathcal{A}(D)$ satisfying $|g(p)| > \sup_{z \in K} |g(z)|$.

The following result is a precise version of Theorem 1.7.

Theorem 6.1. *Assume that S is a Stein manifold, $D \Subset S$ is a strongly pseudoconvex domain with \mathcal{C}^ℓ boundary ($\ell \geq 2$), $r \in \{0, 1, \dots, \ell\}$, and $h: X \rightarrow \bar{D}$ is an $\mathcal{A}_Y^r(D)$ -bundle (Def. 1.6). Choose a distance function d on the manifold $J^r(\bar{D}, X)$ of all r -jets of sections $\bar{D} \rightarrow X$ of h . Let K be a compact $\mathcal{A}(D)$ -convex subset of \bar{D} and let $U \subset S$ be an open set containing K . If the fiber Y enjoys CAP (Def. 1.4) then sections $\bar{D} \rightarrow X$ satisfy the following:*

- (i) *Given a continuous section $f_0: \bar{D} \rightarrow X$ which is of class \mathcal{C}^r on $U \cap \bar{D}$ and holomorphic in $U \cap D$, there exist for every $\epsilon > 0$ an open neighborhood $V \subset U$ of K and a homotopy of sections $f_t: \bar{D} \rightarrow X$ ($t \in [0, 1]$) which are of class \mathcal{C}^r on $V \cap \bar{D}$ and holomorphic on $V \cap D$ such that $f_1 \in \Gamma_{\mathcal{A}}^r(\bar{D}, X)$ and*

$$\sup_{x \in K} d(j_x^r f_t, j_x^r f_0) < \epsilon, \quad t \in [0, 1].$$

- (ii) *Given a homotopy of continuous sections $f_t: \bar{D} \rightarrow X$ ($t \in [0, 1]$) such that $f_0, f_1 \in \Gamma_{\mathcal{A}}^r(\bar{D}, X)$, f_t is holomorphic in $U \cap D$ and of class \mathcal{C}^r on $U \cap \bar{D}$ for each $t \in [0, 1]$, with continuous dependence of $j^r f_t|_{U \cap \bar{D}}$ on t , there are an open neighborhood $V \subset U$ of K and a homotopy of sections $g_{t,s}: \bar{D} \rightarrow X$ ($t, s \in [0, 1]$) satisfying*
 - (1) $g_{t,0} = f_t$, $g_{0,s} = f_0$, $g_{1,s} = f_1$ for all $t, s \in [0, 1]$,
 - (2) $g_{t,1} \in \Gamma_{\mathcal{A}}^r(\bar{D}, X)$ for all $t \in [0, 1]$,
 - (3) $g_{t,s}$ is of class \mathcal{C}^r on $V \cap \bar{D}$, holomorphic on $V \cap D$, and

$$\sup_{x \in K} d(j_x^r g_{t,s}, j_x^r f_t) < \epsilon, \quad s, t \in [0, 1].$$

Proof. The proof of (i) will consist of two parts. In the first part we find a section g_1 of X satisfying (i) on a strongly pseudoconvex domain $\bar{D}_1 \subset \bar{D}$ containing K and such that \bar{D} is obtained from \bar{D}_1 by attaching finitely many convex bumps. In part 2 we show how to approximately extend the solution over each bump to get a solution f_1 on all of \bar{D} .

Part 1. We have $D = \{\rho < 0\}$ where ρ is a \mathcal{C}^ℓ strongly plurisubharmonic function in a neighborhood of \bar{D} , with $d\rho \neq 0$ on $bD = \rho^{-1}(0)$.

Consider first the case when $K \subset D$. Choose a constant $c_1 < 0$ close to 0 such that $K \subset D_1 := \{z \in D: \rho(z) < c_1\}$ and $d\rho \neq 0$ on $\{c_1 \leq \rho \leq 0\} = \bar{D} \setminus D_1$. Since the fiber Y enjoys CAP, the Oka principle on open Stein manifolds [13, Theorem 1.2] gives a holomorphic section g_1 of X over an open neighborhood of \bar{D}_1 which satisfies the conclusion (i) over \bar{D}_1 (with f_1 replaced by g_1). Now proceed directly to Part 2 below.

The situation is more complicated when $K \cap bD \neq \emptyset$. Let $D_0 = \{\rho < c_0\}$ for some $c_0 < 0$ close to 0 such that $d\rho \neq 0$ on $\{c_0 \leq \rho \leq 0\}$. If c_0 is chosen sufficiently close to 0, there is a strongly pseudoconvex domain $B_0 \subset D$

such that $K \subset \bar{B}_0$, (B_0, D_0) is a Cartan pair of class \mathcal{C}^ℓ , and $\bar{B}_0 \cap \bar{D}_0$ is holomorphically convex in D . (Here we used the assumption that K , and hence $K \cap \bar{D}_0$, is $\mathcal{A}(D)$ -convex.) Set $D_1 = D_0 \cup B_0$ (Fig. 2). Suitable choices of D_0 and B_0 insure that \bar{D} is obtained from \bar{D}_1 by attaching finitely many small convex bumps [24, Lemma 12.3].

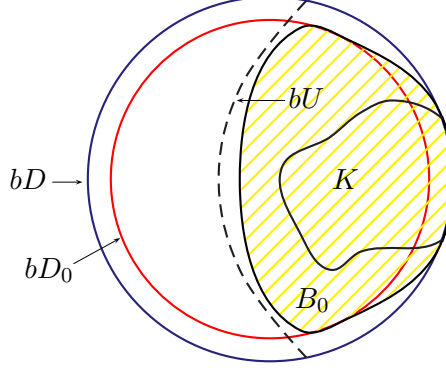


FIGURE 2. The sets K , D_0 , B_0 and U

Let U be as in the statement of the theorem. Choose a strongly pseudoconvex domain $D' \subset D \cap U$ such that $\bar{B}_0 \subset D' \cup bD$. By Corollary 4.2 there exist a domain $P_0 \subset \mathbb{C}^N$ containing the origin and a dominating h -spray $f: \bar{D}' \times P_0 \rightarrow X$ of class $\mathcal{A}^r(D')$ with the central section $f_0|_{\bar{D}'}$.

Let $P \Subset P_0$ be a ball centered at the origin in \mathbb{C}^N . Since the set $(\bar{B}_0 \cap \bar{D}_0) \times \bar{P}$ is holomorphically convex in $D \times P_0$ and Y enjoys CAP, the Oka principle for open Stein manifolds [13, Theorem 1.2] gives an open neighborhood $W \subset D$ of \bar{D}_0 and a holomorphic spray of sections $f': W \times P_0 \rightarrow X$ which approximates f uniformly on a neighborhood of $(\bar{B}_0 \cap \bar{D}_0) \times \bar{P}$; by the Cauchy estimates the approximation is then \mathcal{C}^r close on the latter set. Now we apply Proposition 2.4 to glue the spray f (over \bar{B}_0) and the spray f' (over \bar{D}_0) to a new spray $g: \bar{D}_1 \times P' \rightarrow X$ of class $\mathcal{A}^r(D_1)$ which approximates f in the \mathcal{C}^r topology on $(\bar{B}_0 \cap \bar{D}_0) \times P'$. Its central section $g_1 = g(\cdot, 0): \bar{D}_1 \rightarrow X$ then satisfies condition (i) over \bar{D}_1 .

Part 2: By the choice of D_1 there exist strongly pseudoconvex domains $D_1 \subset D_2 \subset \dots \subset D_m = D$ with \mathcal{C}^ℓ boundaries such that for every $j = 1, \dots, m-1$ we have $D_{j+1} = D_j \cup B_j$, where B_j is a convex bump on D_j (Def. 2.3). Each of the sets B_1, \dots, B_{m-1} is chosen so small that the fiber bundle X is trivial over its closure. To find a solution $f_1: \bar{D} \rightarrow X$ we successively extend the section $g_1: \bar{D}_1 \rightarrow X$ found in Part 1 over each bump by using a finite induction; this will produce sections $g_j \in \Gamma_{\mathcal{A}}^r(D_j, X)$ ($j = 2, 3, \dots, m$) such that the restriction of g_j to \bar{D}_{j-1} is homotopic to g_{j-1} , and is close to g_{j-1} in $\Gamma_{\mathcal{A}}^r(\bar{D}_{j-1}, X)$. Since all steps are of the same kind, it suffices to explain how to obtain g_2 from g_1 .

Corollary 4.2 furnishes a ball $P \subset \mathbb{C}^N$ containing the origin and a dominating h -spray $G_1: \bar{D}_1 \times P \rightarrow X$ of class $\mathcal{A}^r(D_1)$ with the central section g_1 . Let $P' \Subset P$ be a smaller ball around the origin. Since the fiber bundle X is trivial over \bar{B}_1 , we can identify sections of X over (subsets of) \bar{B}_1 with maps to the fiber Y . Also, using local holomorphic coordinates in a neighborhood of \bar{B}_1 in S we shall identify $\bar{B}_1 \cap \bar{D}_1 \subset \bar{B}_1$ with compact convex sets in \mathbb{C}^n .

Using these identifications, we first approximate G_1 in the \mathcal{C}^r topology on $(\bar{B}_1 \cap \bar{D}_1) \times P'$ by a holomorphic map from an open neighborhood of this set (in $\mathbb{C}^n \times \mathbb{C}^N$) to Y (just precompose G_1 with linear contractions to an interior point of its domain); by using the CAP property of Y we then approximate this map by an entire map $\mathbb{C}^n \times \mathbb{C}^N \rightarrow Y$. Passing back to subsets of S , the above procedure gives a spray $G'_1: \bar{B}_1 \times P' \rightarrow X$ of class $\mathcal{A}^r(B_1)$ which approximates G_1 as close as desired in the \mathcal{C}^r topology on $(\bar{D}_1 \cap \bar{B}_1) \times P'$. Finally we glue G_1 and G'_1 into a spray $G_2: \bar{D}_2 \times P' \rightarrow X$ of class $\mathcal{A}^r(D_2)$ by appealing to Proposition 2.4. Letting $g_2 = G_2(\cdot, 0): \bar{D}_2 \rightarrow X$ be the central section of G_2 we complete the induction step.

After m steps of this kind we obtain a section $g_m \in \Gamma_{\mathcal{A}}^r(D, X)$ (which we now call f_1) satisfying the conclusion of Theorem 6.1 (i). A homotopy from f_0 to f_1 with the stated properties is obtained by combining the homotopies obtained in the individual steps of the proof.

Part (ii) is proved in essentially the same way by using the equivalence between CAP and the one-parametric Oka property [13, §5]. A one-parametric spray with the given center is furnished by the parametric part of Corollary 4.2, and we glue sprays by the parametric version of Proposition 2.4. \square

In conclusion we mention the following (apparently) open problem.

Problem 6.2. Assume that D is a smoothly bounded strongly pseudoconvex domain in a Stein manifold, $r \in \mathbb{N} \cup \{\infty\}$, and $X \rightarrow \bar{D}$ is a fiber bundle of class $\mathcal{A}^r(D)$. Let $f: \bar{D} \rightarrow X$ be a section of class $\mathcal{A}(D)$. Is it possible to approximate f uniformly on \bar{D} by sections of class $\mathcal{A}^r(D)$?

Added in the final revision. In the subsequent paper [14] of the second author the conclusion of Theorem 1.1 is extended to Hölder and Sobolev spaces of holomorphic maps, using the fact that the graph of any map $f: \bar{D} \rightarrow Y$ of class $\mathcal{A}(D)$ admits a basis of open Stein neighborhoods in the ambient manifold $S \times Y$ [14, Theorem 1.2]. However, open Stein neighborhoods cannot be used in the proofs of Theorems 1.7 and 6.1 since the bundle is only defined over \bar{D} .

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